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# NILPOTENCY IN GROUP THEORY AND TOPOLOGY

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## 0. Introduction

Our object in these lectures is to indicate an aspect of the interplay between algebra and topology. This aspect is through the application of the notion of nilpotency, applied to groups and to modules over grouprings, to homotopy theory. We start with a study of nilpotency in group theory but, even here, the lines of development are heavily influenced by the homotopy-theoretical applications we have in mind; in this respect, we proceed very much according to standard strategic principles of applied mathematics. However the link between the algebra and the topology is, in a key respect, closer than that between a mathematical model and the 'real world' situation being modeled, in that we can obtain results in homotopy theory which may be interpreted as generalizations of our algebraic results. Thus it is even possible to prove algebraic results from homotopy-theoretical results, so that we can carry out 'applications' in both directions. It will not be possible to give extensive examples of this process here, but it should at least be clear how the localization theorems of Section 3 could, in fact, lead to (rather than follow!) the establishment of a localization theory for nilpotent groups.

Apart from this important notion of application within mathematics itself, the three key features of mathematical methodology which we stress are generalization, relativization and reasoning by analogy.

As to generalization, this process is, of course, familiar to all mathematicians. It is an art, in the sense that there is no unique choice of generalization of a given concept - the criteria determining the validity of a given generalization reside in a subtle blend of the scope of the generalization and the availability of significant theorems analysing the generalized concept. Generalization is also an art in the sense that there can be no algorithmic rule determining when it is appropriate to generalize and which collection of familiar concepts should be subsumed in a common generalization. The satisfaction of the following principles is clearly necessary (and just as clearly not sufficient) to justify a given generalization: (a) the 'collection of familiar concepts' should be bigger than a singleton set, (b) theorems in the generalized context should cast light on special cases where the assertions contained in those theorems were hitherto unknown. Our own generalizations, in these lectures, are, of course, conditioned by the requirement that they contribute to the interplay of algebra and topology.

It is, of course, true that relativization is a special case of generalization - in other words, 'generalization' is a generalization of 'relativization'! Nevertheless, it does seem to deserve explicit mention since it is a common or, as one might dare to say, a standard type of generalization. The classical method of relativizing was to pass from a single object $X$ to a pair of objects $(X,Y)$, where $Y$ is a subobject of $X$; moreover, it might be necessary to impose some special condition on $Y$ as a subobject. We give an example of this type of relativization in connection with Definition 1.1 where we consider a pair $(N,N')$ consisting of a group $N$ and a normal subgroup $N'$ of $N$. Other important examples consist of a topological space $X$ and a closed subspace $Y$ of $X$; and a manifold $M$ and its boundary $\partial M$. However, the categorical point of view suggests that we should not confine ourselves to pairs of objects $(X,Y)$ in which $Y$ is a subobject of $X$. Rather we should broaden the concept of relativization as follows. Given any category $\mathfrak{C}$, we form the category $\mathfrak{C}_r$ of $\mathfrak{C}$-morphisms; thus an object of $\mathfrak{C}_r$ is a morphism $f : Y \to X$ of $\mathfrak{C}$, and a morphism of $\mathfrak{C}_r$, from $f$ to $f'$ is a pair of morphisms $(g,h)$ such that the diagram

$$\begin{tikzcd} Y \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, "h"] \\ Y' \arrow[r, "f'"] & X' \end{tikzcd}$$

commutes. Composition of morphisms in $\mathfrak{C}_r$ is defined in the obvious way. Then a relativization of $\mathfrak{C}$ is a full subcategory of $\mathfrak{C}_r$. This definition is implicitly brought into play in Section 3 when we relativize the notion of nilpotent space to obtain the notion of nilpotent fibre map; since the notion of nilpotent space is a homotopy (rather than a homology) notion, the 'principles' of Eckmann-Hilton duality dictate that, in relativizing, we pass from spaces to fibre maps. This relativization determines our notion of a <u>relative group</u> in Section 2.

In talking of <u>reasoning by analogy</u> we do not intend to convey the impression that we use analogy to achieve mathematical proof - although we would also not wish to deny the possibility of doing so in a specific mathematical context. Here, in these lectures, we confine ourselves to the elaboration of a situation in which we employ intuition and experience to suggest that an idea, taken from a certain mathematical situation, might prove fruitful, if intelligently interpreted, in a somewhat different situation. This type of reasoning is, of course, of the very stuff of rational behavior, and we owe to René Thom the observation that a principal defect of an elementary mathematics education based on elementary set theory (Venn diagrams) is precisely that it is bound to ignore reasoning by analogy. We also owe to Thom the exciting possibility of building reasoning by analogy itself on the foundation of mathematical analysis.*)

To summarize the content of these lectures: in Section 1 we discuss nilpotency in group theory and we recall how we may establish a localization theory for nilpotent groups; in Section 2 we relativize these concepts and results; in Section 3, we discuss the localization of nilpotent spaces and nilpotent maps; and in Section 4 we take up a topic hinted at in [HMR] and give the elegant criterion due to V. Rao [R] for the nilpotency of a mapping cone.

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*) "All these points show what limitations are set by set theory for the description of the usual thought processes. Our usual thinking depends on deep psychic mechanisms, as for example 'analogy', which cannot be reduced to set-theoretical operations. An important role is played in such cases by the organizing isomorphism between semantic fields which are connected by homology with each other." (my translation) R. Thom, 'Moderne' Mathematik - ein erzieherischer und philosophischer Irrtum?, Mathematiker über die Mathematik, Springer (1974), p. 388.

## 1. Concept of nilpotence in group theory

Let $N$ be a group. We define the *lower central series* of $N$ by the rule

$$\Gamma^1 N = N,\ \Gamma^{i+1} N = [N, \Gamma^i N],\ i \geq 1; \tag{1.1}$$

and we say that $N$ is *nilpotent of class* $\leq c$, and write $\text{nil } N \leq c$, if $\Gamma^{c+1} N = \{1\}$. Thus the concept of nilpotent group *generalizes* that of commutative group: $N$ is commutative if and only if $\text{nil } N \leq 1$. We write $\mathbb{N}$ for the category of nilpotent groups, $\mathbb{N}_c$ for the full subcategory of groups $N$ with $\text{nil } N \leq c$.

Examples (a) Let $N = N(p) = \{a,b \mid a^p = b^p = [a,b]\}$, where $p$ is a fixed prime number. It is easy to see that the centre, $ZN$, of $N$ coincides with the commutator subgroup $[N,N] = \Gamma^2 N$, and is cyclic of order $p$. For $[a,b]^p = [a,b^p] = 1$. Moreover $N/ZN$ is generated by the residue classes $\bar{a}$, $\bar{b}$ of $a$, $b$ and $\bar{a}^p = \bar{b}^p = 1$. Thus we have a *central* extension

$$\mathbb{Z}/p \rightarrowtail N(p) \twoheadrightarrow \mathbb{Z}/p \times \mathbb{Z}/p, \tag{1.2}$$

showing that $|N| = p^3$ and $\text{nil } N = 2$. If $p = 2$, $N$ is the celebrated *quaternionic* group of order $8$.

(b) Let $F = F(x_\alpha)$ be the free group on the symbols $(x_\alpha)$ and let $N(x_\alpha) = F/\Gamma^{c+1}F$. Then $N(x_\alpha)$ is the *free nilpotent group of class* $c$ *on the symbols* $(x_\alpha)$. Every group in $\mathbb{N}_c$ is the homomorphic image of $N(x_\alpha)$, for some suitable choice of $(x_\alpha)$. This example also shows that there are groups of arbitrary nilpotency—hardly surprising!

Our first main result generalizes the observation, based on (1.2) in Example (a), that $N(p) \in \mathbb{N}_2$.

Theorem 1.1. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be a central extension of groups*

*(that is,* $N'$ *is in the center of* $N$ *and* $N/N' = N''$*). Then*[*)]
$\mathrm{nil}\, N'' \le \mathrm{nil}\, N \le \mathrm{nil}\, N'' + 1$.

Proof. That $\mathrm{nil}\, N'' \le \mathrm{nil}\, N$ is obvious from (1.1) and requires no hypothesis of centrality. On the other hand, if $\Gamma^{c+1}N'' = \{1\}$, then $\Gamma^{c+1}N \subseteq N'$ and $\Gamma^{c+2}N \subseteq [N,N'] = \{1\}$, since $N'$ is central.

We now seek to *generalize* Theorem 1.1. We must bear in mind that, given an extension of groups $N' \rightarrowtail N \twoheadrightarrow N''$, we cannot infer the nilpotence of $N$ from that of $N'$ and $N''$ (the converse implication, on the other hand, obviously holds). For let $N = S_3$, the symmetric group on 3 symbols, 1, 2, 3. If $x$ is the cyclic permutation (123), then $x$ generates a normal subgroup $N' = \mathbb{Z}/3$ and $N'' = N/N' = \mathbb{Z}/2$. Thus $N'$ and $N''$ are nilpotent--indeed, commutative--but $N$ is not nilpotent. For an easy calculation shows that $\Gamma^i N = N'$, $i \ge 2$. Thus our generalization cannot consist of simply discarding the centrality condition in Theorem 1.1; we must weaken it judiciously. We are indeed led to the following *relativization* of the concept of nilpotency.

Definition 1.1. Let $N'$ be normal in $N$, written $N' \triangleleft N$. Then the *(relative) lower central series of* $N'$ *in* $N$ is given by

$$\Gamma_N^1 N' = N',\ \Gamma_N^{i+1} N' = [N, \Gamma_N^i N'],\ i \ge 1. \tag{1.3}$$

We say that the embedding of $N'$ in $N$ is *nilpotent of class* $\le c$ and write $\mathrm{nil}_N N' \le c$ if $\Gamma_N^{c+1} N' = \{1\}$.

Notice that we have simultaneously *relativized* nilpotency $(\Gamma_N^i N = \Gamma^i N)$ and *generalized* centrality ($N'$ is central in $N$ if and only if $\mathrm{nil}_N N' \le 1$). Notice also that each $\Gamma_N^i N'$ is normal in $N$. We now generalize Theorem 1.1.

Theorem 1.1g. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of groups. Then*

[*)] Our result remains valid if we adopt the convention, as we will henceforth, that $N$ not nilpotent $\Rightarrow \mathrm{nil}\, N = \infty$.

$$\max(\mathrm{nil}\, N'', \mathrm{nil}_N N') \le \mathrm{nil}\, N \le \mathrm{nil}\, N'' + \mathrm{nil}_N N'.$$

Proof. We easily generalize the argument of Theorem 1.1. In particular, if $\Gamma^{c+1}N'' = \{1\}$, $\Gamma_N^{d+1}N' = \{1\}$, then $\Gamma^{c+1}N \subseteq N' = \Gamma_N^1 N'$, $\Gamma_N^{c+2}N \subseteq \Gamma_N^2 N'$, ..., $\Gamma^{c+d+1}N \subseteq \Gamma_N^{d+1}N' = \{1\}$.

Let $G$ be a group. In the theory of $G$-modules, there is also a notion very much akin to that of the lower central series. Indeed, if $A$ is a $G$-module, we define the *lower central* $G$-*series* of $A$ by the rule

$$\Gamma_G^1 A = A, \ \Gamma_G^{i+1} A = \mathrm{gp}(a - xa), \ x \in G, \ a \in \Gamma_G^i A, \ i \ge 1; \tag{1.4}$$

and we say that $A$ is $G$-*nilpotent of class* $\le c$, written $\mathrm{nil}_G A \le c$, if $\Gamma_G^{c+1} A = \{0\}$. Notice that each $\Gamma_G^i A$ is a *submodule* of $A$, and that

$$\Gamma_G^{i+1} A = \Gamma_G^i(\Gamma_G^1 A). \tag{1.5}$$

Just as nilpotency generalized commutativity $(c = 1)$, so here the case $c = 1$ is the case of *trivial* action of $G$ on $A$. This brings the ideas of nilpotency and $G$-nilpotency very close, since a commutative group is precisely a group $N$ such that the action of $N$ on itself by conjugation is trivial.

As we shall see in Section 3, the two concepts of nilpotent group and $G$-nilpotent module are the essential ingredients in our *application* of nilpotency to topology. Here we pursue our programme of generalizing our concepts, motivated by the desire to find a useful *common* generalization of the two concepts just mentioned; of course, we should select such a generalization to include also relative nilpotency as expressed in Definition 1.1.

This last remark provides the clue. For if $N' \triangleleft N$ then $N$ operates on $N'$ by conjugation; thus we may hope to find a fruitful generalization of Definition 1.1 and of the lower central $G$-series of a $G$-module by supposing $N$ to be a $G$-group, that is, a group on which the group $G$ acts, and defining a lower central $G$-series of $N$.

Definition 1.2. Let $N$ be a G-group. We define the *lower central G-series* of $N$ by the rule

(1.6) $\Gamma_G^1 N = N$, $\Gamma_G^{i+1}N = gp(a.xb.a^{-1}b^{-1})$, $a \in N$, $b \in \Gamma_G^i N, x \in G$, $i \geq 1$;

and we say that $N$ is *G-nilpotent of class* $\leq c$, written $nil_G N \leq c$, if $\Gamma_G^{c+1}N = \{1\}$.

It is immediately obvious that this definition coincides with (1.4) above if $N$ is commutative. It is also easy to see that Definition 1.2 generalizes Definition 1.1. For suppose $N \triangleleft G$ and let $G$ operate on $N$ by conjugation. We then have, from Definitions 1.1 and 1.2, two definitions of $\Gamma_G^i N$, and to see that they coincide, it suffices to verify that if $M \triangleleft G$ and if $K = gp(axbx^{-1}a^{-1}b^{-1})$, $a \in N$, $b \in M$, $x \in G$, then $K = [G,M]$.

The following remarks are also pertinent: (a) if $N$ is G-nilpotent it is certainly nilpotent (as a group) and $nil\, N \leq nil_G N$; (b) each $\Gamma_G^i N$, in (1.6), is a normal G-closed subgroup of $N$; (c) $\Gamma_G^{i+1}N = gp([N,\Gamma_G^i N], xb.b^{-1})$, $x \in G$, $b \in \Gamma_G^i N$; (d) (1.5) does *not* generalize—on the other hand, we still have the inequality $\Gamma_G^i(\Gamma_G^1 N) \subseteq \Gamma_G^{i+1}N$, so that we may infer

(1.7) $$nil_G \Gamma_G^1 N \leq nil_G N - 1.$$

The relation (1.7) is, naturally, very useful in fashioning proofs by induction on G-nilpotency class.

A more surprising observation is that not only can Definition 1.1 be subsumed under Definition 1.2, but also the other way round! For let $N$ be a G-group. We form the *semidirect product* of $N$ and $G$; thus $P = N \rtimes G$ is defined as follows. The underlying set of $P$ is the cartesian product of the underlying sets of $N$ and $G$, and the group operation in $P$ is given by

(1.8) $$(a_1,x_1)(a_2,x_2) = (a_1.x_1a_2,x_1x_2).$$

There is an obvious embedding $N \rightarrowtail P$ and a projection $P \twoheadrightarrow G$, giving rise to a group extension

$$N \rightarrowtail P \twoheadrightarrow G \tag{1.9}$$

which *splits on the right* in the sense that there is a section homomorphism $G \to P$ (the obvious embedding). We then have

Theorem 1.3. $$\Gamma_P^i N = \Gamma_G^i N.$$

Proof. We argue by induction on $i$, the case $i = 1$ being trivial. Now if we conjugate $a \in N$ by $(c,x) \in P$, $c \in N$, $x \in G$, we obtain $c.xa.c^{-1}$. Thus, let us assume $\Gamma_P^i N = \Gamma_G^i N$, for some $i \geq 1$. The preceding remark, together with Definition 1.1, immediately shows that a system of generators of $\Gamma_P^{i+1} N$ consists of elements of the form $a.xb.a^{-1}b^{-1}$, $a \in N$, $b \in \Gamma_G^i N$, $x \in G$, establishing the inductive hypothesis and the theorem.

We next draw an immediate consequence from Theorems 1.1g and 1.3.

Corollary 1.4. *Let $N$ be a G-group and let $P$ be the semidirect product of $N$ and $G$. Then $P$ is nilpotent if and only if $G$ is nilpotent and $N$ is G-nilpotent. Indeed,*

$$\max(\mathrm{nil}\, G, \mathrm{nil}_G N) \leq \mathrm{nil}\, P \leq \mathrm{nil}\, G + \mathrm{nil}_G N.$$

We close this section with a theorem which shows that the G-nilpotence of a group $N$ is, in a very strict sense, determined by the nilpotence of $N$ and the G-nilpotence of $N_{ab}$, the abelianization of $N$. It is thus a generalization of a very apt kind of the two nilpotency concepts which led to its formulation.

Theorem 1.5. *Let $N$ be a G-group. Then $N$ is G-nilpotent if and only if $N$ is a nilpotent group and $N_{ab}$ is G-nilpotent.*

Proof. The entire argument is essentially due to Derek Robinson [R], although he considered a very slightly different situation. Certainly if

$N$ is $G$-nilpotent it is nilpotent; and, just as certainly, if $N$ is $G$-nilpotent then any $G$-homomorphic image, and so in particular $N_{ab}$, is $G$-nilpotent, Conversely, let $N_{ab}$ be $G$-nilpotent. It is easy to see that then $\otimes^i N_{ab}$, the $i$-fold tensor power of $N_{ab}$, with diagonal action, is also $G$-nilpotent. But $\Gamma^i N/\Gamma^{i+1}N$, as a $G$-homomorphic image of $\otimes^i N_{ab}$, is also $G$-nilpotent. Since $N$ is nilpotent and

$$(1.10) \qquad \Gamma^i N/\Gamma^{i+1}N \rightarrowtail N/\Gamma^{i+1}N \twoheadrightarrow N/\Gamma^i N$$

is a central extension of $G$-groups, Theorem 1.5 follows from (1.10) by induction on $i$, using the following easy *generalization* of Theorem 1.1.

Theorem 1.1g'. *Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be a central extension of* $G$*-groups. Then*

$$\max(\mathrm{nil}_G N'', \mathrm{nil}_G N') \leq \mathrm{nil}_G N \leq \mathrm{nil}_G N'' + \mathrm{nil}_G N' .$$

We will now formulate a common generalization of Theorems 1.1g and 1.1g'.

Let $N' \rightarrowtail N \twoheadrightarrow N''$ be an extension of $G$-groups and let $P$ be the semidirect product of $N$ and $G$. Now $N$ and $G$ both act on $N'$, with $N$ acting by conjugation and the actions are related by the rule

$$(1.11) \qquad x(a.b) = xa.xb\ ,\ x \in G\ ,\ a \in N\ ,\ b \in N' .$$

For $x(a.b) = x(aba^{-1}) = (xa)(xb)(xa^{-1}) = xa.xb$ .

We may then prove

Theorem 1.6. Let $N, N'$ be $G$-groups and let $N$ act on $N'$. Let $P$ be the semidirect product of $N$ and $G$. Then there exists an action of $P$ on $N'$ extending the given actions of $N$ and $G$ if and only if (1.11) is satisfied. Moreover this action, given by

(1.12) $\qquad (a,x)b = a.xb\ ,\quad x \in G\ ,\ a \in N\ ,\ b \in N'$

*is unique*.

*Proof*. We will be content to show that (1.12) *is* a group action if and only if (1.11) is satisfied. For we have

$$(a_1,x_1)((a_2,x_2)b) = (a_1,x_1)(a_2.x_2b) = a_1.x_1(a_2.x_2b)\ ,$$

$$((a_1,x_1)(a_2,x_2))b = (a_1(x_1a_2),x_1x_2)b = a_1(x_1a_2).x_1x_2b =$$
$$= a_1.(x_1a_2.x_1x_2b)\ ,$$

so that $(a_1,x_1)((a_2,x_2)b) = ((a_1,x_1)(a_2,x_2))b$ for all $a_i \in N$, $x_i \in G$, $i = 1,2$, $b \in N'$ if and only if (1.11) holds.

Thus if $N' \rightarrowtail N \twoheadrightarrow N''$ is an extension of G-groups, we may regard $N'$ as a P-group, and the common generalization we seek is the following.

*Theorem 1.7. Let* $N' \rightarrowtail N \twoheadrightarrow N''$ *be an extension of G-groups. Then*

$$\max(\mathrm{nil}_G N'', \mathrm{nil}_P N') \leq \mathrm{nil}_G N \leq \mathrm{nil}_G N'' + \mathrm{nil}_P N'\ .$$

That Theorem 1.7 does generalize Theorem 1.1g' may be seen from the observation that, if $N'$ is central in $N$ then the action of $N$ on $N'$ is trivial so that $\mathrm{nil}_P N' = \mathrm{nil}_G N'$. More generally we have

*Theorem 1.8. Let* $N,N'$ *be G-groups. Then the trivial action of* $N$ *on* $N'$ *yields* (1.11) *and the induced action of* $P$ *on* $N'$ *is just the action via the projection* $P \twoheadrightarrow G$. *For this action we have* $\Gamma_P^i N' = \Gamma_G^i N'$.

The crucial role played by the relation (1.11) in our discussion may be reflected in the definition that *the G-group $N$ acts on the G-group $N'$* precisely when $N,N'$ are G-groups and the group $N$ acts on the group $N'$ so that (1.11) is satisfied. We can in this way generalize nilpotent group theory to *nilpotent G-group theory.*

A useful result in connexion with Theorem 1.7 is the following [HRS] .

Theorem 1.9. Let the G-group $N$ act on the G-group $N'$ and let $P = N \rtimes G$ . Then $N'$ is P-nilpotent if and only if $N'$ is N-nilpotent and G-nilpotent.

Here we will merely observe that a crucial role is played in the proof by Theorem 1.5 which allows us to assume $N'$ abelian.

We close this section by recalling from [HMR] certain key properties of the P-localization of nilpotent groups. We work in the category $\mathfrak{N}$ of nilpotent groups and we describe a group $M$ in $\mathfrak{N}$ as P-local, where $P$ is a family of primes*), provided the function $M \to M$, given by $x \mapsto x^q$ , $x \in M$ , is bijective for $q$ outside $P$ .
We then say that, for a group $N$ in $\mathfrak{N}$ , the homomorphism $e: N \to N_P$ in $\mathfrak{N}$ P-localizes $N$ provided that $N_P$ is P-local and that, for any $\varphi : N \to M$ in $\mathfrak{N}$ with $M$ P-local, there exists a unique $\psi : N_P \to M$ with $\psi e = \varphi$ . Plainly if such a homomorphism $e$ exists it is unique up to canonical isomorphism.

Theorem 1.10 [HMR] Every group in $\mathfrak{N}$ may be P-localized. Moreover $N_P \in \mathfrak{N}_c$ if $N \in \mathfrak{N}_c$ .
Note that Theorem 1.10 implies that the localization theory in $\mathfrak{N}$ extends the fairly elementary localization theory in $\mathfrak{N}_1$ , the category of abelian groups.

A further crucial result is

Theorem 1.11. Localization is exact.

In order to be able easily to handle questions related to localization it is necessary to have a means of recognizing the localizing homomorphism. To this end we make the following definitions.

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*) Note that $P$ is now no longer a semidirect product!

Definition 1.3. Let $\varphi : N \to M$ be a homomorphism of nilpotent groups. Let $P$ be a family of primes and let $P'$ be the semigroup (with identity) generated by the complement of $P$. Then

(i) $\varphi$ is P-injective if $\ker \varphi$ is a $P'$-torsion group;

(ii) $\varphi$ is P-surjective if, for all $y \in M$, there exists $n \in P'$ with $y^n \in \mathrm{im}\, \varphi$;

(iii) $\varphi$ is P-bijective if it is both P-injective and P-surjective.

We then have [HMR]

Theorem 1.12. Given $\varphi : N \to M$ in $\mathfrak{N}$, then

(i) $\varphi$ is P-injective if and only if $\varphi_P : N_P \to M_P$ is injective;

(ii) $\varphi$ is P-surjective if and only if $\varphi_P : N_P \to M_P$ is surjective

Theorem 1.13. (Recognition principle) Given $\varphi : N \to M$ in $\mathfrak{N}$, then $\varphi$ P-localizes $N$ if and only if $M$ is P-local and $\varphi$ is P-bijective.

As an example of how Theorem 1.13 may be used, let us prove

Theorem 1.14. Let $N$ be nilpotent, $H, K$ subgroups of $N$. Then $(H \cap K)_P = H_P \cap K_P$.

Proof. We know, from Theorem 1.11, that $H_P$, $K_P$ are subgroups of $N_P$. Certainly $e : N \to N_P$ induces

$$e_o : H \cap K \to H_P \cap K_P ,$$

so it remains to prove

(a) $H_P \cap K_P$ admits $q^{th}$ roots, $q \in P'$;

(b) if $y, z \in H_P \cap K_P$ and $y^q = z^q$, $q \in P'$, then $y = z$;

(c) $e_o$ is P-injective;

(d) $e_o$ is P-surjective.

Now (b) and (c) are trivial, being inherited from $e : N \to N_P$. As to (a), let $y \in H_P \cap K_P$. Since $y \in H_P$, there exists $a \in H_P$ with $y = a^q$; and since $y \in K_P$, there exists $b \in K_P$ with $y = b^q$.

Thus $a^q = b^q$ and, since $q^{th}$ roots are unique in $N_P$, $a = b$. Thus $y = a^q$ with $a \in H_P \cap K_P$.

To prove (d), let $y \in H_P \cap K_P$. Then $\exists m \in P'$ with $y^m = ea$, $a \in H$; and $\exists n \in P'$ with $y^n = eb$, $b \in K$. It follows that $ea^n = eb^m$, so that $a^n = b^m z$, with $z$ a $P'$-torsion element of $N$. It may then be shown that, if $z^\ell = 1$, $\ell \in P'$, and if nil $N \leq c$, then $a^{n\ell^c} = b^{m\ell^c}$. It follows that $a^{n\ell^c} \in H \cap K$ and $y^{mn\ell^c} \in \operatorname{im} e_0$, establishing (d).

## 2. Relative groups

We define a <u>relative group</u> to be a group extension

$$N \rightarrowtail G \xrightarrow{\kappa} Q . \tag{2.1}$$

Note that, if we apply the Eilenberg-MacLane functor $K(-,1)$, we convert a relative group into a fibration. Thus, following our remarks in the introduction, we will be particularly interested in the cases where $N$ is nilpotent (we then say that (2.1) is a <u>weakly nilpotent</u> relative group) or where $G$ acts nilpotently on $N$ by conjugation (we then say that (2.1) is a <u>strongly nilpotent</u> relative group). By analogy with homotopy theory, we will designate (2.1) simply by means of the projection $\kappa$, and describe $\kappa$ itself as a relative group.

Our first result relates to the construction of relative groups. Given the diagram

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "\alpha"] & G \arrow[r, two heads, "\kappa"] & Q \\ M & & \end{tikzcd} \tag{2.2}$$

it is natural to ask whether (2.2) can be embedded in a diagram of group extensions

$$(2.3) \qquad \begin{tikzcd} N \arrow[r, tail] \arrow[d, "\alpha"'] & G \arrow[r, "\varepsilon"] \arrow[d, "\beta"] & Q \arrow[d, equal] \\ M \arrow[r, tail] & H \arrow[r, "\lambda"] & Q \end{tikzcd}$$

Now the homomorphism $\alpha : N \to M$ determines an action of $N$ on $M$ by the rule

$$(2.4) \qquad a \cdot b = (\alpha a) b (\alpha a^{-1}) \quad , \; a \in N \; , \; b \in M \; .$$

We now prove

<u>Theorem 2.1</u>. <u>We may embed</u> (2.2) <u>in a full diagram</u> (2.3) <u>if and only if there is an action of</u> $G$ <u>on</u> $M$ , <u>extending</u> (2.4) <u>and satisfying</u>

$$(2.5) \qquad x \cdot \alpha a = \alpha(xax^{-1}) \quad , \quad x \in G \; , \quad a \in N$$

<u>Proof</u>. If the diagram (2.3) exists we may define an action of $G$ on $M$ by the rule

$$(2.6) \qquad x \cdot b = (\beta x) b (\beta x^{-1}) \quad , \quad x \in G \; , \; b \in M \; .$$

It is then obvious that this action extends (2.4) and satisfies (2.5).

Conversely, suppose we are given an action of $G$ on $M$, extending (2.4) and satisfying (2.5). Form $S = M \rtimes G$, the semidirect product of $M$ and $G$, and map $N$ to $S$ by the function $a \mapsto (\alpha a^{-1}, a)$ , $a \in N$. It follows from (2.4) that this is a homomorphism. It is plainly injective and, in fact, maps $N$ onto a normal subgroup $\tilde{N}$ of $S$. For we may show, using (2.4) and (2.5), that

$$(b,x)(\alpha a^{-1},a)(b,x)^{-1} = (\alpha(xa^{-1}x^{-1}), xax^{-1}) \quad , \; a \in N \; , \; b \in M \; , \; x \in G \; .$$

Set $H = S/\tilde{N}$ and write $\{b,x\}$ for the element of $H$ containing $(b,x) \in S$. It is easy to see that $b \mapsto \{b,1\}$ embeds $M$ in $H$ as a normal subgroup. We define $\lambda : H \to Q$ by $\lambda\{b,x\} = \varepsilon x$. It is plain that $\lambda$ is well-defined, and that $\lambda$ is surjective with kernel the image of $M$. Finally we define $\beta : G \to H$ by $\beta x = \{1,x\}$, and clearly we have achieved the full diagram.

Remarks. (i) It is easy to see that we have constructed, by the argument above, a full diagram in which (2.6) holds. This will be important in considering the uniqueness question.

(ii) It is plain that we cannot always embed (2.2). A simple counterexample may be constructed using the following observation. If (2.3) exists, then $\ker\alpha = \ker\beta$. Now suppose given $N \rightarrowtail G \twoheadrightarrow Q$ and a subgroup $N_1$ of $N$ which is normal in $N$ but not in $G$. Then if $\alpha$ projects $N$ onto $M = N/N_1$, we cannot construct (2.3). Such an example is provided by taking $G$ to be the dihedral group, as follows

$$G = \{x, a, a' \mid x^2 = a^2 = a'^2 = 1,\ aa' = a'a,\ xa = ax,\ xa' = aa'x\}$$

$$N = \{a, a'\},\ N_1 = \{a'\},\ \alpha \text{ projects } N \text{ onto } M = N/N_1 .$$

Plainly, if $\alpha$ is surjective, the condition that $\ker\alpha$ be normal in $G$ is both necessary and sufficient for the embedding. Indeed we then define the action of $G$ on $M$ by (2.5). In general, this condition is not sufficient; indeed, there are counterexamples with $\alpha$ injective.

We now take up the question of the uniqueness of (2.3). Obviously we do not wish to distinguish between (2.3) and

(2.3')
$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "\alpha"'] & G \arrow[r, two heads, "\kappa"] \arrow[d, "\beta'"] & Q \arrow[d, equal] \\ M \arrow[r, tail] & H' \arrow[r, two heads, "\lambda'"] & Q \end{tikzcd}$$

if there is a homomorphism $\omega : H \to H'$ (which is then necessarily an isomorphism) such that $\omega\beta = \beta'$ and the diagram

(2.7)
$$\begin{tikzcd} M \arrow[r, tail] \arrow[d, equal] & H \arrow[r, two heads, "\lambda"] \arrow[d, "\omega"] & Q \arrow[d, equal] \\ M \arrow[r, tail] & H' \arrow[r, two heads, "\lambda'"] & Q \end{tikzcd}$$

commutes. We prove uniqueness as a corollary of the following universal property of (2.3).

*Theorem 2.2. Given (2.3), constructed as in the proof of Theorem 2.1, and the associated action (2.6) of $G$ on $M$, given $\varrho : M \to M_0$, and given the commutative diagram*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \xrightarrow{\kappa} & Q \\ \downarrow{\scriptstyle\alpha_0} & & \downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\delta} \\ M_0 & \rightarrowtail & K & \xrightarrow{\tau} & R \end{array} \qquad \alpha_0 = \varrho\alpha \ ,$$

*there exists a unique homomorphism $\omega : H \to K$ satisfying $\omega\beta = \gamma$ and rendering commutative the diagram*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \xrightarrow{\kappa} & Q \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \| \\ M & \rightarrowtail & H & \xrightarrow{\lambda} & Q \\ \downarrow{\scriptstyle\varrho} & & \downarrow{\scriptstyle\omega} & & \downarrow{\scriptstyle\delta} \\ M_0 & \rightarrowtail & K & \xrightarrow{\tau} & R \end{array}$$

*if and only if*

(2.8) $$\varrho(x\cdot b) = (\gamma x)\varrho b(\gamma x^{-1}) \ , \quad x \in G \ , \ b \in M \ .$$

*Proof*. Define $\omega : H \to K$ by $\omega\{b,x\} = (\varrho b)(\gamma x)$, $b \in M$, $x \in G$. We observe that $\omega$ is well-defined, and we use (2.8) to show that it is a homomorphism. Plainly $\omega|M = \varrho$ and $\omega\beta = \gamma$; and $\omega$ is uniquely determined by these two conditions. Finally,

$$\tau\omega\{b,x\} = \tau((\varrho b)(\gamma x)) = \tau\gamma x = \delta\kappa x = \delta\lambda\{b,x\} \ ,$$

completing the proof of the existence and uniqueness of $\omega$. The converse follows from the observation that $\omega$, as defined, is only a homomorphism if (2.8) holds.

*Corollary 2.3. Let*

$$\begin{array}{ccccc} N & \rightarrowtail & G & \xrightarrow{\kappa} & Q \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta'} & & \| \\ M & \rightarrowtail & H' & \xrightarrow{\lambda'} & Q \end{array}$$

be a diagram such that the associated action of $G$ on $M$, given by

$$x\cdot b = (\beta' x)b(\beta' x^{-1}) \ , \quad x \in G \ , \ b \in M \ ,$$

coincides with (2.6). Then there exists an isomorphism $\omega : H \to H'$ such that $\omega\beta = \beta'$ and (2.7) commutes.

Remark. No further uniqueness could possibly be expected. For without demanding that the extension respect the given action, we do not even determine $H$ up to isomorphism. Thus consider any action $\theta$ of $Q$ on $N$ and let $H_\theta$ be the associated semidirect product. Then we have the diagram

$$\begin{tikzcd} 1 \arrow[r, tail] \arrow[d] & Q \arrow[r, two heads] \arrow[d] & Q \arrow[d, equal] \\ N \arrow[r, tail] & H_\theta \arrow[r, two heads] & Q \end{tikzcd} ,$$

and the bottom row is certainly not determined by the groups $N$ and $Q$.

We confine ourselves henceforth to weakly nilpotent relative groups $\kappa$, that is, to group extensions

$$N \rightarrowtail G \xrightarrow{\kappa} Q$$

with $N$ nilpotent. Let us P-localize $N$; thus we obtain

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "e_P"'] & G \arrow[r, two heads, "\kappa"] & Q \\ N_P & & \end{tikzcd}$$

and we are ready to apply our previous results. Plainly the conjugation action of $G$ on $N$ 'extends' uniquely to an action of $G$ on $N_P$ in the sense that the diagram (we write $e$ for $e_P$ if no confusion need be feared)

(2.9)
$$\begin{tikzcd} N \arrow[r, "x"] \arrow[d, "e"'] & N \arrow[d, "e"] \\ N_P \arrow[r, "x"] & N_P \end{tikzcd}$$

commutes for all $x \in G$. This action extends the action of $N$ on $N_P$. For if we define $a : N_P \to N_P$ by $a \cdot b = (ea)b(ea^{-1})$, then the diagram

(2.10)
$$\begin{tikzcd} N \arrow[r, "a"] \arrow[d, "e"'] & N \arrow[d, "e"] \\ N_P \arrow[r, "a"] & N_P \end{tikzcd}$$

commutes; and there is only one action of $N$ on $N_P$ making (2.10) commute for all $N$. Also the condition corresponding to (2.5) is precisely the commutativity of (2.9). Thus Theorem 2.1 allows us to infer the existence of a diagram

(2.11)
$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "e"'] & G \arrow[r, two heads, "\kappa"] \arrow[d, "\bar{e}"'] & Q \arrow[d, equal] \\ N_P \arrow[r, tail] & G_{(P)} \arrow[r, two heads, "\bar{\kappa}"] & Q \end{tikzcd} .$$

Note that it is not necessary here to specify the action of $G$ on $N_P$, since it is uniquely determined by the requirement that it satisfy the equivalent of (2.5), that is, that (2.9) commute for all $x \in G$.

We call $\bar{\kappa}$ the P-<u>localization</u> of $\kappa$. To justify this term, we must show that it possesses the universal property corresponding to that in the absolute case. We call a (weakly nilpotent) relative group $\tau$ <u>P-local</u> if $\ker \tau$ is P-local, so that $\bar{\kappa}$ <u>is</u> P-local and we suppose given a map from $\kappa$ to $\tau$, with $\tau$ P-local,

(2.12)
$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "\alpha"] & G \arrow[r, two heads, "\kappa"] \arrow[d, "\beta"] & Q \arrow[d, "\gamma"] \\ M \arrow[r, tail] & K \arrow[r, two heads, "\tau"] & R \end{tikzcd} .$$

We may then factorize $\alpha$ as $\alpha = \varrho e$ with $\varrho : N_P \to M$ uniquely determined. We wish to show that there is a unique factorization of (2.12) as

(2.13)
$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "e"] & G \arrow[r, two heads, "\kappa"] \arrow[d, "\bar{e}"] & Q \arrow[d, equal] \\ N_P \arrow[r, tail] \arrow[d, "\varrho"] & G_{(P)} \arrow[r, two heads, "\bar{\kappa}"] \arrow[d, "\omega"] & Q \arrow[d, "\gamma"] \\ M \arrow[r, tail] & K \arrow[r, two heads, "\tau"] & R \end{tikzcd} .$$

According to Theorem 2.2, we must show that

$$(2.14) \qquad \varrho(x \cdot b) = (\beta x)\varrho b(\beta x^{-1}), \; x \in G, \; b \in N_P .$$

We consider the diagram, for arbitrary $x \in G$, extending (2.9),

$$(2.15) \qquad \begin{tikzcd} N \arrow[r, "x"] \arrow[d, "e"'] & N \arrow[d, "e"] \\ N_P \arrow[r, "x"] \arrow[d, "\varrho"'] & N_P \arrow[d, "\varrho"] \\ M \arrow[r, "x"] & M \end{tikzcd} .$$

In this diagram, the bottom row is conjugation with $\beta x$. Thus (2.14) asserts the commutativity of the bottom square of (2.15). We know that the top square in (2.15) commutes, and, since $\varrho e = \alpha = \beta | N$, we know that the composite square of (2.15) commutes. Thus $x\varrho e = \varrho x e : N \to M$; but then uniqueness implies that $x\varrho = \varrho x : N_P \to M$, yielding (2.14). This shows that we have indeed uniquely factored our map $\kappa \to \tau$ through $\bar{\kappa}$, so that we have achieved the P-localization of the relative group $\kappa$. We note that, by the uniqueness of the action of $G$ on $N_P$ extending the conjugacy action of $G$ on $N$, it follows that, given any map

$$\begin{tikzcd} N \arrow[r, tail] \arrow[d, "e"'] & G \arrow[r, two heads, "\kappa"] \arrow[d, "f"] & Q \arrow[d, equal] \\ N_P \arrow[r, tail] & H \arrow[r, two heads, "\lambda"] & Q \end{tikzcd}$$

then $\lambda$ is the P-localization of $\kappa$.

We close this section by showing, by an example, where we need the concept of strong nilpotency. We first quote a key result which we will not prove here.

We say that $\kappa$ is <u>strongly nilpotent</u> if the embedding of $N$ in $G$ is nilpotent (Definition 1.1). The result we need, however, only requires weak nilpotency.

*Theorem 2.4.* [H] *If* $\kappa$ *is weakly nilpotent, then* $(\Gamma_G^i N)_P = \Gamma^i_{G_{(P)}} N_P$ .

Now consider the diagram, and its P-localisation * ,

$$
(2.16)\qquad
\begin{tikzcd}
N \arrow[r, tail] \arrow[d, "\alpha"] & G \arrow[r, two heads, "\kappa"] \arrow[d, "\beta"] & Q \arrow[d, equal] \\
M \arrow[r, tail] & H \arrow[r, two heads] & Q
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
N_P \arrow[r, tail] \arrow[d, "\alpha_P"] & G_{(P)} \arrow[r, two heads, "\bar{\kappa}"] \arrow[d, "\beta_{(P)}"] & Q \arrow[d, equal] \\
M_P \arrow[r, tail] & H_{(P)} \arrow[r, two heads, "\bar{\lambda}"] & Q
\end{tikzcd}
$$

with $\kappa, \lambda$ weakly nilpotent. By Theorem 1.12 and some elementary reasoning, we see that

(2.17) $\beta$ P-surjective $\Rightarrow$ $\alpha$ P-surjective $\Leftrightarrow$ $\alpha_P$ surjective $\Leftrightarrow$ $\beta_{(P)}$ surjective.

However, in general, we cannot infer the P-surjectivity of $\beta$ from that of $\alpha$ as Example 2.1 will show. We may, on the other hand, prove

*Theorem 2.5. If* $\lambda$ *is strongly nilpotent, then* $\alpha$ *P-surjective* $\Rightarrow$ $\beta$ *P-surjective.*

*Proof.* Let $y \in H$ . Then plainly $y = (\beta x)u$ for some $x \in G$ , $u \in M$ . Assume inductively that

$$y^n = (\beta x)u \text{ , for some } n \text{ prime to } P \text{ , } x \in G \text{ , } u \in \Gamma^i_H M \text{ .}$$

Since $\alpha$ is P-surjective, there exists $m$ prime to $P$ with $u^m = \alpha a$ , $a \in N$ . Then

$$y^{mn} = ((\beta x)u)^m \equiv \beta x^m \alpha a = \beta(x^m a) \bmod [H, \Gamma^i_H M] \text{ ,}$$

so that

$$y^{mn} = (\beta x_1)u_1 \text{ , } x_1 \in G \text{ , } u_1 \in \Gamma^{i+1}_H M \text{ .}$$

This establishes the inductive hypothesis; by taking $i$ sufficiently large, we infer that $\beta$ is P-surjective.

---

* We deduce the existence and uniqueness of $\beta_{(P)}$ from Theorem 2.2 exactly as for $\omega$ in (2.13).

Corollary 2.6. *If* $\kappa$ *is strongly nilpotent, then* $\bar{e}: G \to G_{(P)}$ *is* *P-surjective.*

Proof. By Theorem 2.4 it follows that $\bar{\kappa}$ is strongly nilpotent if $\kappa$ is strongly nilpotent. But $e: N \to N_P$ is P-surjective.

We may make another interesting deduction from Theorem 3.3.

Corollary 2.7. *Let* $\alpha$ *in (2.16) induce* $\alpha^i: \Gamma_G^i N \to \Gamma_H^i M$ . *Then if* $\alpha$ *is P-surjective, so is* $\alpha^i$ .

Proof. We consider the commutative diagram

$$\begin{array}{ccc} \Gamma_G^i N & \xrightarrow{e} & \Gamma_{G(P)}^i N_P \\ \downarrow{\scriptstyle \alpha^i} & & \downarrow{\scriptstyle \alpha_P^i} \\ \Gamma_H^i M & \xrightarrow{e} & \Gamma_{H_{(P)}}^i M_P \end{array} ,$$

where the labeling of the horizontal rows is justified by Theorem 2.4. Since $\alpha$ is P-surjective, it follows from (2.17) that $\alpha_P$ and $\beta_{(P)}$ are surjective and hence (by an easy induction on $i$ ) $\alpha_P^i$ is surjective. Thus $\alpha^i$ is P-surjective by Theorem 1.12.

To show that we need to assume $\kappa$ strongly nilpotent in Corollary (2.6) consider the following example.

Example 2.1. Let $N = Z$ , written additively; let $Q = Z/2 = (x)$ act on $N$ by $xa = -a$ . Let $G = N \rtimes Q$ . Let $P = (2)$ . Then $N_2$ is the group of rational numbers representable as fractions $\frac{m}{n}$ with $n$ odd. Moreover $G_{(2)}$ is again the semidirect product $N_2 \rtimes Q$ , where $Q$ acts on $N_2$ by $xb = -b$ . We show that $\bar{e} : G \to G_{(2)}$ is not 2-surjective. For consider the element $(b,x) \in G_{(2)}$ . Then $(b,x)^2 = (b+xb, x^2) = (0,1)$ , so that $(b,x)^q = (b,x)$ for any odd exponent $q$ . Thus if $b$ is not in the image of $e : N \to N_2$ , that is, if $b$ is not an integer, $(b,x)^q$ is not in the image of $\bar{e}$ for any odd $q$, so that $\bar{e}$ is certainly not 2-surjective.

This example was discovered independently by Urs Stammbach.

## 3. Homotopy theory of nilpotent spaces

In this section we very briefly review the homotopy theory of nilpotent spaces, with special reference to definitions and results from the previous two sections. See [HMR] for further details.

Let $\mathfrak{H}_o$ be the homotopy category of pointed connected spaces of the pointed homotopy type of a CW-complex. For any such space $X$ there is defined an action of $\pi_1 X$, the fundamental group, on the higher homotopy groups $\pi_n X$, $n \geq 2$.

Definition 3.1. The space $X$ in $\mathfrak{H}_o$ is nilpotent if $\pi_1 X$ is nilpotent and acts nilpotently on the higher homotopy groups $\pi_n X$.

Examples (a) If $X$ is 1-connected it is obviously nilpotent.

(b) $X$ in $\mathfrak{H}_o$ is called simple if $\pi_1 X$ is commutative and acts trivially on $\pi_n X$, $n \geq 2$. Simple spaces are plainly nilpotent; in particular, topological groups and H-spaces in $\mathfrak{H}_o$ are nilpotent.

(c) If $G$ is a nilpotent Lie group, then the classifying space $BG$ is a nilpotent space.

(d) Let $W$ be a compact polyhedron, let $X$ be a nilpotent space, and let $X^W$ $(X^W_{fr})$ be the space of pointed (free) maps of $W$ into $X$. Then each component of $X^W$ $(X^W_{fr})$ is nilpotent; indeed, if $W$ is connected, the nilpotency of each component of $X^W$ follows without assuming $X$ nilpotent.

Example (d) shows how naturally nilpotent spaces arise even if one's interest is confined to the homotopy theory of 1-connected spaces.

A key property of 1-connected spaces, which facilitates many arguments and constructions, is that their Postnikov tower consists of principal fibrations. Thus, given $X$ in $\mathcal{S}_o$, we may construct a series of fibrations

(3.1) $$p_n : X_n \to X_{n-1}, \quad n = 1,2,\ldots$$

such that

(a) $X_o$ is the base point;

(b) $p_n$ is a fibration with fibre the Eilenberg-MacLane space $K(\pi_n X, n)$;

(c) there are maps $q_n : X \to X_n$ such that $q_n$ is an n-equivalence and $p_n q_n = q_{n-1}$, $n = 1,2,\ldots$ .

We call (3.1) the Postnikov tower of $X$ and we have the

Theorem 3.1. *If $X$ is 1-connected, then $p_n : X_n \to X_{n-1}$ is induced by a (classifying) map $X_{n-1} \xrightarrow{g_n} K(\pi_n X, n+1)$.* Such an induced fibration is called principal.

Now let us suppose that $X$ is in $\mathcal{S}_o$ and that $\pi = \pi_1 X$ acts nilpotently on $A = \pi_n X$, $n \geqslant 2$. Indeed, let us assume, in the notation of Section 1, that $\mathrm{nil}_\pi A \leq c$ so that $\Gamma_\pi^{c+1} A = \{0\}$. We may then prove

Theorem 3.2. *With the given notation $\Gamma_\pi^{c+1} A = \{0\}$ if and only if we may factor $p_n : X_n \to X_{n-1}$ as a composition of fibrations*

(3.2) $$X_n = Y_c \to Y_{c-1} \to \cdots \to Y_i \xrightarrow{u_i} Y_{i-1} \to \cdots \to Y_1 \to Y_o = X_{n-1},$$

*where $u_i$ is a principal fibration induced by a map $v_i : Y_{i-1} \to K(\Gamma^i A/\Gamma^{i-1} A, n+1)$, $i = 1,2,\ldots,c$ .*

There is a very similar theorem, which the reader can supply, relating to the nilpotency of $\pi$ and the fibration $X_1 \to pt$ . We thus have the

<u>Corollary 3.3</u>. <u>The space</u> $X$ <u>in</u> $\mathfrak{H}_0$ <u>is simple if and only if its Postnikov tower consists of principal fibrations</u>.

However, our main concern here is to point out that we may generalize many theorems from 1-connected spaces to nilpotent spaces by using the refined Postnikov tower, consisting of the principal fibrations $u_i$ of (3.2).

In particular, we may use the characterization of nilpotent spaces by means of Theorem 3.2 to establish a <u>localization theory</u> in the full subcategory $\mathfrak{N}\mathfrak{H}$ of $\mathfrak{H}_0$ , consisting of nilpotent spaces.

Let $X$ be in $\mathfrak{N}\mathfrak{H}$ . We say that $X$ is <u>P-local</u>, where $P$ is a family of primes, if each $\pi_n X$ , $n \geq 1$ , is P-local. Exploitation of the refined Postnikov tower enables us to prove that this condition is equivalent to asking that the homology groups $H_n X$ be P-local, $n \geq 1$ . We say that $e : X \to X_P$ in $\mathfrak{N}\mathfrak{H}$ <u>P-localizes</u> $X$ if $X_P$ is P-local and if, for all $f : X \to Y$ in $\mathfrak{N}\mathfrak{H}$ with $Y$ P-local, there exists a unique $g : X_P \to Y$ with $ge = f$ . Notice that this definition refers to the <u>homotopy</u> category $\mathfrak{N}\mathfrak{H}$ ; thus uniqueness and commutativity of diagrams are only asserted up to homotopy. Indeed, the construction itself is to be understood as a homotopy construction. In fact, we prove two main theorems.

<u>Theorem 3.3</u>. <u>Any</u> $X$ <u>in</u> $\mathfrak{N}\mathfrak{H}$ <u>may be P-localized</u>.

<u>Theorem 3.4</u>. <u>Let</u> $f : X \to Y$ <u>in</u> $\mathfrak{N}\mathfrak{H}$ . <u>Then the following assertions are equivalent</u>:

(i) $f$ P-localizes;

(ii) $\pi_n f : \pi_n X \to \pi_n Y$ <u>P-localizes for every</u> $n \geq 1$ ;

(iii) $H_n f : H_n X \to H_n Y$ <u>P-localizes for every</u> $n \geq 1$ .

The proof heavily exploits the refined Postnikov tower. Within the category $\mathfrak{S}_1$ of 1-connected spaces (so that $\mathfrak{S}_1$ is a full subcategory of $\mathfrak{N}\mathfrak{S}$), it is possible to carry out a construction of $X_P$ which is simpler both conceptually and practically. In this construction we use the cellular structure of $X$ as a CW-complex. Such a procedure is satisfactory in $\mathfrak{S}_1$ because we may assume that we always remain in $\mathfrak{S}_1$ as we attach cells to build up $X$ (it is always legitimate to choose, within the homotopy type of a 1-connected space $X$, a model space such that $X^0$ is the base point and $X^2$ is a bunch of 2-spheres; We then imitate the cellular structure of $X$ by building up $X_P$ by means of P-local cells, that is, cones on P-localized spheres. However, such a procedure does not work for a nilpotent space since, as we build up a nilpotent space $X$ by attaching cells we constantly pass in and out of $\mathfrak{N}\mathfrak{S}$. For example, in the natural cellular structure of $X = \mathbb{R}P(n)$, it is natural to take $X^m = \mathbb{R}P(m)$, $m \leq n$. But $\mathbb{R}P(m)$ is nilpotent if and only if $m$ is odd. Thus we need the procedure via the refined Postnikov tower described in [HMR]. We will take up again this problem of the 'bad behaviour' of $\mathfrak{N}\mathfrak{S}$ with regard to attaching cells in the next section.

Meanwhile we consider here the appropriate relativization of the theory so far outlined. First the relativization of a pointed connected space, in homotopy theory, should be a fibration $p : E \to B$ of pointed connected spaces such that the fibre $F$ is connected; this last condition is equivalent to the requirement that $p$ induce a surjection of fundamental groups. Notice that we recover the notion of a pointed connected space by taking $B$ to be a point. We then have an evident homotopy category of which such fibrations are the objects and we must now consider when such an object deserves to be described as nilpotent.

Two possible definitions emerge: we describe $p$ as weakly nilpotent if $F$ is nilpotent, and as strongly nilpotent if the action of $\pi_1 E$ on $\pi_n F$ is nilpotent for all $n \geq 1$. These definitions each provide a relativization of the notion of nilpotency contained in Definition 3.1; and they generalize the notions of the same name described in Section 2. For a group extension (relative group) $N \rightarrowtail G \xrightarrow{\kappa} Q$ may be realized by a fibre map $K(N,1) \to K(G,1) \xrightarrow{p} K(Q,1)$ of Eilenberg. MacLane spaces and $\kappa$ is (weakly, strongly) nilpotent if and only if $p$ is (weakly, strongly) nilpotent.

We may relativize the notion of the Postnikov tower of a space, to obtain the Moore-Postnikov tower of a fibration. We may then relativize the characteristic property of nilpotent spaces by proving that $p$ is strongly nilpotent if, and only if, its Moore-Postnikov tower admits a principal refinement.

The next step would then be to introduce P-localization. As suggested by the previous observation, this can be done very much along the lines of the construction in $\mathfrak{N}\mathfrak{H}$, provided that $p$ is strongly nilpotent. The result of P-localizing the fibration $F \to E \xrightarrow{p} B$ would then be a fibration $F_P \to E_{(p)} \xrightarrow{\bar{p}} B$ and this would have an appropriate universal property. Moreover, the localizing map $\bar{e}: E \to E_{(p)}$ could be recognized by the fact that it P-localizes the homotopy groups of $p$. The localization of weakly nilpotent fibrations presents additional difficulties which can probably be overcome by using the powerful techniques of [BK]. It would, however, be much more satisfactory to proceed entirely within the (relative) homotopy category described, exploiting the localization of weakly nilpotent relative groups of the preceding section.

## 4. Mapping cones and nilpotency

A fundamental fact about nilpotent spaces is expressed by the following result.

*Theorem 4.1. Let* $F \to E \xrightarrow{p} B$ *be a fibration with all spaces connected. Then* $F$ *is nilpotent if* $E$ *is nilpotent.*

It has long been realized

that this result does not dualize in the sense of Eckmann-Hilton (nor, indeed, would one expect it to). Thus let $f : X \to Y$ be a map and let $Z$ be the mapping cone (cofibre) of $f$. Then it is easy to construct examples where $Y$ is nilpotent but $Z$ is not. For example if $f : S^1 \to S^1$ has degree 2 then $Z$ is $RP(2)$ and so definitely not nilpotent. A remarkable result due to Vidhyanath Rao [R] shows just how rarely $Z$ inherits nilpotency from* $Y$.

*Theorem 4.2* [R]. *Let* $f : X \to Y$ *be a map with* $Y$ *nilpotent and* $X$ *connected, and let* $Z$ *be the mapping cone of* $f$. *Then* $Z$ *is nilpotent if and only if one of the following conditions is satisfied*:

(a) $f_*$ *is a surjection of* $\pi_1 X$ *onto* $\pi_1 Y$ ;

(b) $X$ *is homologically trivial*;

(c) *there exists a prime* $p$ *such that* $\pi_1 Z$ *is a finite* $p$-*group and each* $H_n X$ , $n \geq 1$ , *is a* $p$-*group of finite exponent*.

Before proving this, we point out the following consequence.

*Corollary 4.3. Let* $Y$ *be nilpotent but not 1-connected and let* $n \geq 3$ . *Then* $Y \cup e^n$ *is not nilpotent*.

Thus we cannot construct a nilpotent space of dimension $\geq 3$ which is not 1-connected by attaching cells and remain always within the nilpotent category.

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* A similar result, but involving hypotheses of finite type, had been obtained by R.H. Lewis [L1,2].

We now prove Rao's theorem; our proof will differ a little from that of Rao. We divide the argument into a topological part and an algebraic part. The topological part consists of proving that $Z$ is nilpotent if and only if $Z\pi \otimes H_n X$ is a nilpotent $\pi$-module for all $\geq 1$, where $\pi = \pi_1 Z$. To show this, we first replace $f$, as we may by an inclusion (cofibration) $X \subseteq Y$ so that $Z = Y/X$. Let $\tilde{Z}$ be the universal cover of $Z$. Then, by pulling back, we obtain the diagram

$$\begin{array}{ccccc} \pi \times X & \subseteq & \tilde{Y} & \to & \tilde{Z} \\ \downarrow & & \downarrow & & \downarrow \\ X & \subseteq & Y & \to & Z \end{array} .$$

Note that $\tilde{Y}$ is the regular covering of $Y$ with cover transformation group $\pi$. Now $\tilde{Z}$ is obtained from $\tilde{Y}$ by identifying each $\xi \times X$ to a point $z_\xi$, $\xi \in \pi$. Thus $\tilde{Y}/\pi \times X$ is obtained from $\tilde{Z}$ by identifying the discrete set of points $\{z_\xi\}$ to a single point. It follows that the map $\tilde{Y} \to \tilde{Z}$ induces an isomorphism

$$H_n(\tilde{Y}, \pi \times X) \cong H_n\tilde{Z} \quad , \quad n \geq 2 .$$

Thus the homology sequence of the pair $(\tilde{Y}, \pi \times X)$ may be transcribed as

$$(4.1) \quad \cdots \to Z\pi \otimes H_n X \to H_n\tilde{Y} \to H_n\tilde{Z} \to Z\pi \otimes H_{n-1}X \to \cdots \to H_2\tilde{Y} \to H_2\tilde{Z} \to$$
$$\to Z\pi \otimes H_1 X \to H_1\tilde{Y} .$$

Moroeover, (4.1) is a sequence of $\pi$-modules. Now $\pi$ is certainly nilpotent as a quotient of $\pi_1 Y$. Thus $Z$ is nilpotent if and only if $\pi$ acts nilpotently on $H_n\tilde{Z}$, $n \geq 2$. Since $Y$ is nilpotent it is easy to see that $\pi$ acts nilpotently on $H_n\tilde{Y}$, $n \geq 1$. It therefore follows from (4.1) that $Z$ is nilpotent if and only if $\pi$ acts nilpotently on $Z\pi \otimes H_n X$, $n \geq 1$.

There are two trivial cases in which this will occur. First, it will occur if $\pi = \{1\}$ . However, $\pi_1 Y$ being nilpotent, this occurs precisely when $f_*$ maps $\pi_1 X$ onto $\pi_1 Y$ , that is, in case (a) of Theorem 4.2. Second, it will occur if $H_n X = \{0\}$ , $n \geq 1$ , which is case (b) of Theorem 4.2. Thus our proof is complete when we have established the following proposition.

Proposition 4.4. Let $\pi$ be a non-trivial group and $A$ a non-trivial abelian group. Then $Z\pi \otimes A$ is a nilpotent $\pi$-module if and only if there exists a prime $p$ such that $\pi$ is a finite p-group and $p^n A = 0$ for some positive integer $n$.

Proof. We appeal to the following well-known facts:

(i) $Z\pi$ is never a nilpotent $\pi$-module;

(ii) $(Z/p)\pi$ is nilpotent if and only if $\pi$ is a finite p-group.

Now suppose that $\pi$ is a finite p-group and $p^n A = 0$ . Consider the exact sequence

$$p^i A \rightarrowtail p^{i-1} A \twoheadrightarrow p^{i-1} A / p^i A \ , \quad i = 0,1,\ldots,n \ .$$

Since $p^{i-1}A/p^iA$ is a vector space over $Z/p$ , it follows from fact (ii) that $Z\pi \otimes p^{i-1}A/p^iA$ is a nilpotent $\pi$-module. Since $Z\pi$ is free as an abelian group, we have an exact sequence of $\pi$-modules

$$(4.2) \quad Z\pi \otimes p^i A \rightarrowtail Z\pi \otimes p^{i-1} A \twoheadrightarrow Z\pi \otimes p^{i-1} A / p^i A \ , \ i = 0,1,\ldots,n \ ,$$

and we use (4.2) and downward induction on $i$ to infer that $Z\pi \otimes A$ is a nilpotent $\pi$-module.

Conversely, suppose that $\mathrm{nil}_\pi Z\pi \otimes A = c$ . If $A$ had an element of infinite order, then (since $Z\pi$ is free abelian) $Z\pi \otimes A$ would contain $Z\pi$ as a submodule and fact (i) would be contradicted. Let $A$ have an element of order $p$ . Then $Z\pi \otimes A$ contains $Z\pi \otimes Z/p = Z/p(\pi)$ as a submodule and fact (ii) tells us that $\pi$ is a finite p-group. It follows that there exists a prime $p$ such that $\pi$ is a finite p-group and $A$ is a p-torsion group. It remains to show that the p-exponent of $A$ is finite.

To this end, suppose that $A$ possesses an element of order $p^m$. Then $Z/p^m \subseteq A$ and $C_p \subseteq \pi$, where $C_p$ is the (multiplicative) cyclic group of order $p$. Since $ZC_p$ is a direct summand in $Z\pi$ we have inclusions

$$ZC_p \otimes Z/p^m \subseteq Z\pi \otimes Z/p^m \subseteq Z\pi \otimes A \;,$$

where the first inclusion is an inclusion of $C_p$-modules and the second is an inclusion of $\pi$-modules. It follows that

$$\text{(4.3)} \qquad \operatorname{nil}_{C_p} Z/p^m(C_p) \leq c \quad .$$

We now conclude the proof with the following computation, due to Urs Stammbach.

<u>Proposition 4.5</u>. $\operatorname{nil}_{C_p} Z/p^m(C_p) = m(p-1)+1$ .

<u>Proof of Proposition 4.5</u>. Let $J$ be the augmentation ideal of $R = Z/p^m(C_p)$ . Plainly

$$|R| = p^{mp} \;, \quad |R/J| = p^m \;, \quad |J/J^2| = p \;.$$

The last result follows from the fact that $J/J^2 \cong Z/p \otimes Z/p^m$ . Now it is easy to see that the product map

$$J/J^2 \otimes J^{i-1}/J^i \to J^i/J^{i+1} \quad , \quad i = 2,3,\ldots$$

is surjective. Thus, for $i = 2,3,\ldots$ , $|J^i/J^{i+1}| = p$ or $1$ . Let $k$ be the smallest value of $i$ such that $|J^i/J^{i+1}| = 1$ . Since we know from fact (ii) that $|J^i| = 1$ for some $i$ , it follows that $|J^{k-1}| = p$ , $J^k = 1$ , so $k = \operatorname{nil}_{C_p} R$ . On the other hand we have $|R| = |R/J|\,|J/J^2| \ldots |J^{k-1}/J^k|$ , so $p^{mp} = p^{m+k-1}$ , whence

$$k = m(p-1) + 1 \;.$$

From Proposition 4.5 and (4.3) we infer that $m \leq [\frac{c-1}{p-1}]$ , so that $A$ has finite p-exponent; and Proposition 4.4 and Theorem 4.2 are proved.

It would clearly be desirable to generalize Theorem 4.2 to a study of homotopy pushouts; this generalization is currently being undertaken by V. Rao. It would also be interesting to generalize Proposition 4.5 to obtain the $C_{p^\ell}$-nilpotency of $Z/p^m(C_{p^\ell})$.

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